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AN EXTENDED ABEL-JACOBI MAP

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Abstract. We solve the problem of inversion of an extended Abel-Jacobi map

$$\int_{P_0}^{P_1} \omega + \cdots + \int_{P_0}^{P_{g+n-1}} \omega = \mathbf{z}, \qquad \int_{P_0}^{P_1} \Omega_{j1} + \cdots + \int_{P_0}^{P_{g+n-1}} \Omega_{j1} = Z_j, \quad j = 2, \ldots, n,$$

where Ω_{j1} are (normalised) abelian differentials of the third kind. In contrast to the extensions already studied, this one contains meromorphic differentials having a common pole Q_1. This inversion problem arises in algebraic geometric description of monopoles, as well as in the linearization of integrable systems on finite-dimensional unreduced coadjoint orbits on loop algebras.

1. Introduction.

The purpose of this note is to present and invert an extension of the classical Abel-Jacobi theorem that was recently encountered in the construction of nonabelian magnetic monopoles [BE06] and also appeared in the linearization of integrable systems on finite-dimensional unreduced coadjoint orbits on various loop algebras ([AHH93, Gav99, GHHW85, RST79]). We believe this extension is of independent interest and will now describe it, placing it in context.

Let Γ be a Riemann surface of genus g with a canonical homology basis $\{a_i, b_i\}_{i=1}^{g}$ and holomorphic differentials $\omega = (\omega_1, \ldots, \omega_g)$ normalized so that $\oint_{a_i} \omega_j = \delta_{ij}$. (We follow the conventions of [FK80].) Let Λ be the rank $2g$ lattice in $\mathbb{C}^g = (z_1, \ldots, z_g) = \mathbf{z}$ generated by the vectors of periods of ω with respect to $a_1, \ldots, b_g$ and $J(\Gamma) = \mathbb{C}^g/\Lambda$ be the Jacobian of the curve Γ. The Abel map $A(P) = \int_{P_0}^{P} \omega$ gives a well defined holomorphic map of Γ into $J(\Gamma)$ that may be extended to a map of divisors. Abel's theorem identifies the divisor of a meromorphic function on Γ with degree zero divisors in the kernel of A and Jacobi's (inversion) theorem says that every point on the Jacobian $J(\Gamma)$ is expressible as the image under the Abel map of a positive divisor of degree g: that is, for all $\mathbf{z} \in J(\Gamma)$ we may find $P_1, \ldots, P_g \in \Gamma$ such that

$$\int_{P_0}^{P_1} \omega + \cdots + \int_{P_0}^{P_g} \omega = \mathbf{z}.$$

Analytically this inversion makes use of Riemann's theta function. The Abel-Jacobi theorems play a central role in the theory of Riemann surfaces and find important application in the solution of integrable systems. The modern approach to integrability places special attention on algebraically completely integrable systems: systems whose real invariant tori may be extended to complex algebraic tori (Abelian varieties) related to an algebraic curve and whose complex phase flows become straight line motion on the tori. A Lax equation, for example, will frequently result in a curve Γ, the relevant Abelian variety is the Jacobian $J(\Gamma)$ and the dynamics is linearizable on the Jacobian. In this setting the physical solution is obtained by solving the Jacobi inversion problem.

Although many known integrable systems fit into the framework just described, some extensions have been required over the years. In particular, if one wishes to describe the rotation matrices of some integrable tops ([BBEIM94, GZ99, F99]) or to obtain asymptotic (heteroclinic) solutions (such us umbilic geodesic trajectories on a triaxial ellipsoid) ([Er89, F99]), one arrives at quadratures that, in addition to the holomorphic differentials already noted, involve various meromorphic differentials of the third kind, Ω_{P_+,P_-}, with simple poles at $P_\pm$ and having residues ± 1. Clebsch and Gordan [CG86] described an appropriate extension of Abel-Jacobi theory to take these into account. Namely, suppose $X_1, Y_1, \ldots, X_s, Y_s$ are distinct pairs of points on Γ and Ω_{X_i,Y_i} the corresponding differentials of the third kind. We will assume throughout that such differentials are normalized to have vanishing a periods. Then (for $P_0 \notin \{X_1, Y_1, \ldots, X_s, Y_s\}$) the extended Abel map $(\int_{P_0}^P \omega, \int_{P_0}^P \Omega_{X_1,Y_1}, \ldots, \int_{P_0}^P \Omega_{X_s,Y_s})$ admits the inversion of

$$\text{(1.1)}\quad \int_{P_0}^{P_1} \omega + \cdots + \int_{P_0}^{P_{g+s}} \omega = \mathbf{z}, \qquad \int_{P_0}^{P_1} \Omega_{X_i,Y_i} + \cdots + \int_{P_0}^{P_{g+s}} \Omega_{X_i,Y_i} = Z_i, \quad i = 1, \ldots s.$$

In this setting the Jacobian of Γ is replaced by a noncompact Abelian variety, an algebraic group, and generalized theta functions replace the Riemann theta function in the analytic inversion. This result may be viewed as a limit of the classical Abel-Jacobi theory for a genus $g+s$ curve as s a-cycles pinch to zero resulting in a singular surface. (The $s=1$ account of this may be found in [Fay73], for a more general case one can consult [F99, AF01].)

In this paper we consider a special degeneration of the map (1.1), when all the meromorphic differentials have a *common* pole. Namely, let $Q_1, \ldots, Q_n$ be distinct points of our surface Γ and let $\Omega_{21}, \ldots, \Omega_{n1}$ be normalized meromorphic differentials of the third kind (with vanishing a periods) having pairs of simple poles at points $(Q_2, Q_1), \ldots, (Q_n, Q_1)$ respectively, and such that

$$\operatorname{Res}_{Q_2} \Omega_{21} = \cdots = \operatorname{Res}_{Q_n} \Omega_{n1} = 1 \quad \text{and} \quad \operatorname{Res}_{Q_1} \Omega_{21} = \cdots = \operatorname{Res}_{Q_1} \Omega_{n1} = -1.$$

Let, as above, $\mathbf{z} = (z_1, \ldots, z_g) \in \mathbb{C}^g$ and $\hat{\mathbf{z}} = (\mathbf{z}, Z_2, \ldots, Z_n) \in \mathbb{C}^{g+n-1}$. (Note that now there is no Z_1-variable here.) Then (for $P_0 \notin \{Q_1, \ldots, Q_n\}$) the extended Abel map $\mathcal{A}(P) = (\int_{P_0}^P \omega, \int_{P_0}^P \Omega_{21}, \ldots, \int_{P_0}^P \Omega_{n1})$ admits the inversion of

$$\text{(1.2)}\quad \int_{P_0}^{P_1} \omega + \cdots + \int_{P_0}^{P_{g+n-1}} \omega = \mathbf{z}, \qquad \int_{P_0}^{P_1} \Omega_{j1} + \cdots + \int_{P_0}^{P_{g+n-1}} \Omega_{j1} = Z_j, \quad j = 2, \ldots, n.$$

In our extension there is obvious overlap with the Clebsch-Gordan theory for $s = 1$ and $n = 2$. Note that the theta-functional formulas describing the inversion of (1.1) for $s > 1$ do not survive under the limit $Y_1, \ldots, Y_s \to Q_1$; the pinchings needed to view this result as a degeneration of the usual Abel-Jacobi theory are delicate (see also Remark 1 below). Apparently, the inversion problem in this case was not treated in the literature, and we will give a direct solution to this problem rather than consider such a limit.

A precise formulation of our result and the generalized theta functions which arise in the analytical inversion will be given in the next section. Here we remark that the extension (1.2) naturally appears in the linearization of integrable systems on finite-dimensional unreduced coadjoint orbits of various loop algebras (see, among others, [AHH93, Gav99, RST79]). The systems are described by $n \times n$ matrix Lax pairs with a rational parameter, and the spectral curve Γ is an n-fold cover of $\mathbb{P}^1$ with infinite points $\infty_1, \ldots, \infty_n$. The corresponding quadratures involve a complete set of g holomorphic differentials on Γ, as well as differentials with pairs of simple poles $(\infty_1, \infty_2), \ldots, (\infty_1, \infty_n)$. For a generic choice of Darboux coordinates on the unreduced orbits, the quadratures describe evolution of a divisor of $g+n-1$ finite points on Γ and thereby have the form (1.2)[1]

The same extension appears in algebraic completely integrable system associated to the BPS limit of $su(2)$ magnetic monopoles and Γ is the attendant curve, which has a natural spatial interpretation [Hit83]. In common with many such integrable systems, Baker-Akhiezer functions on the curve play a prominent role. Such functions are a slight extension to the class of meromorphic functions that allow essential singularities at a finite number $n \geq 1$ of points; they have many properties similar to those of meromorphic functions. While for a meromorphic function one needs to prescribe $g+1$ poles in the generic situation, a non-trivial Baker-Akhieser function exists with g arbitrarily prescribed poles on a surface of genus g. The construction of such functions may be made rather explicitly in terms of Riemann's theta function and involves an otherwise unspecified generic divisor of degree $g+n-1$. In the monopole setting, a solution to the inversion problem (1.2) means that this unspecified divisor will have no physical consequences: such should be the case, and consequently led to the present theory developed in [BE06, §3.2].

2. Notation and Results

Let $\hat{\Gamma}$ be the $4g$-sided polygon with symbol $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$ obtained by dissecting Γ along the cycles of the chosen homology basis. We assume that our points Q_i do not lie on any of our canonical cycles. Let γ_i be (disjoint) cycles around the points Q_i lying in $\hat{\Gamma}$ oriented so that $\oint_{\gamma_i} \Omega_{i1} = \mathrm{Res}_{Q_i} \Omega_{i1} = 1$ and $\oint_{\gamma_1} \Omega_{i1} = \mathrm{Res}_{Q_1} \Omega_{i1} = -1$ (for $2 \leq i \leq n$). Choose a point of $\partial\hat{\Gamma}$ as a vertex and let $\hat{\Gamma}_c$ denote the domain $\hat{\Gamma}$ with (disjoint) cuts from this vertex to each of the points Q_i. Then $\int_{P_0}^{P} \Omega_{j1}$ is single valued on $\hat{\Gamma}_c$.

The generalized map $\mathcal{A}(P)$ has $2g+n-1$ independent period vectors corresponding to the $2g$ canonical cycles a_i, b_i and cycles $\gamma_2, \ldots, \gamma_n$. The factor of $\mathbb{C}^{g+n-1}$ by the lattice generated by these vectors will be called the generalized Jacobian $\mathrm{Jac}(\Gamma; Q_1, \ldots, Q_n)$.

Introduce (for $n \geq 2$) the *generalized theta function*

$$\begin{aligned}
&\Theta_n(\hat{\mathbf{z}}) \equiv \Theta(\mathbf{z}, Z_2, \ldots, Z_n) \\
&= \sum_{i=2}^{n} \exp(Z_i - \mathcal{K}_i - \Delta_i)\, \theta\left(\mathbf{z} - K - \mathcal{S} + A(Q_i)\right) - \theta\left(\mathbf{z} - K - \mathcal{S} + A(Q_1)\right), \quad (2.1)
\end{aligned}$$

$$\Delta_i = \sum_{k\neq i,\, k\neq 1} \int_{P_0}^{Q_k} \Omega_{i1}, \quad \mathcal{S} = \sum_{i=1}^{n} A(Q_i)\,, \quad (2.2)$$

where, as above, $A(P) = \int_{P_0}^{P} (\omega_1, \ldots, \omega_g)$ is the customary Abel map and $\theta(\mathbf{z})$ is the canonical Riemann theta-function with the Riemann period matrix τ of Γ,

$$\theta(\boldsymbol{z}) \equiv \theta(\boldsymbol{z}; \tau) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp(\imath\pi \boldsymbol{n}^T \tau \boldsymbol{n} + 2\imath\pi \boldsymbol{z}^T \boldsymbol{n}). \quad (2.3)$$

We will suppress throughout the dependence on τ. The vector K in (2.1) is the vector of Riemann constants (see e.g., [Fay73]), while $\mathcal{K}_i$ ($i \geq 2$) are the following constants (fixed in

[1]Note that a special choice of such coordinates fixes $n-1$ points of the divisor at infinity, and the the first equation in (1.2) becomes a standard Abel map admitting the inversion by the Riemann theta-function. This approach was pursued in [AHH93].

our discussion below)

$$\mathcal{K}_i = \sum_{k=1}^{g} \oint_{a_k} \omega_k(P) \int_{P_0}^{P} \Omega_{i1} + \oint_{b_k} \Omega_{i1}. \tag{2.4}$$

Observe that for $n \geq 3$ there is a recursive description

$$\Theta_n(\hat{\mathbf{z}}) = \exp(Z_n - \mathcal{K}_n - \Delta_n)\, \theta \left(\mathbf{z} - K - \sum_{i=1}^{n-1} A(Q_i) \right) + \Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(Q_n)). \tag{2.5}$$

We will establish two inversion theorems based on the function

$$f(P) = \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P)).$$

We first observe

Proposition 2.1. 1). *Although $f(P)$ is not single-valued on the curve Γ, its zeros do not depend on the choice of path in the Abel map $\mathcal{A}(P)$.*

2). *The differential $d \ln f(P)$ is regular at Q_1 and has poles of residue -1 at each of $Q_2, \dots, Q_n$.*

3). *The function $f(P)$ has precisely $g + n - 1$ zeros (possibly with multiplicity).*

With this our generalized inversion theorems are then

Theorem 2.2. *Let $\hat{\mathbf{z}} = (\mathbf{z}, Z_2, \dots, Z_n)$ be such that the function $f(P) = \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P))$ does not vanish identically on $\hat{\Gamma} \setminus \{Q_1, \dots, Q_n\}$. In this case, if $\hat{\mathbf{z}}$ is the right hand side of (1.2), the $g + n - 1$ zeros give a unique solution to the inversion of the extended map (1.2).*

Theorem 2.3. *Let $\hat{\mathbf{z}} \in \{\Theta_n(\hat{\mathbf{z}}) = 0\}$ be such that $f(P) = \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P))$ does not vanish identically on $\hat{\Gamma} \setminus \{Q_1, \dots, Q_n\}$. Then there exists a unique positive divisor $P_1 + \dots + P_{g+n-2}$ such that*

$$\hat{\mathbf{z}} = \mathcal{A}(P_1) + \dots + \mathcal{A}(P_{g+n-2}). \tag{2.6}$$

Remark 1. According to [CG86, F99], the inversion of the generalized map (1.1), that includes meromorphic differentials Ω_{X_i,Y_i} with *different* poles, is given by zeros of the generalized theta-function

$$\tilde{\theta}(\tilde{\mathbf{z}} - \mathcal{A}(P) - \tilde{K}), \qquad \tilde{\mathbf{z}} = (\mathbf{z}, Z_1, \dots, Z_s)^T, \quad \tilde{K} = (K, k_1, \dots, k_s)^T, \quad k_j = \text{const},$$

which is represented by the sum of 2^s terms,

$$\tilde{\theta}(\mathbf{z}, Z) = \sum_{\varepsilon_1, \dots \varepsilon_s = \pm 1} \exp\left(\frac{1}{2}(\varepsilon, Z) + \frac{1}{4}(\varepsilon, \mathbf{S}\varepsilon) \right) \theta \left(\mathbf{z} + \frac{1}{2}\varepsilon_1 q_1 + \dots + \frac{1}{2}\varepsilon_s q_s \right), \tag{2.7}$$

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_s)^T, \quad (\varepsilon, Z) = \sum_{l=1}^{s} \varepsilon_l Z_l, \quad (\varepsilon, \mathbf{S}\varepsilon) = \sum_{l,r=1, l\neq r}^{s} S_{lr} \varepsilon_l \varepsilon_r.$$

where $q_l = (q_{l1}, \dots, q_{lg})^T$ is the b-period vector of $\Omega_{X_l Y_l}$ and S_{lr} are given by

$$q_{li} = \int_{X_l}^{Y_l} \bar{\omega}_i = \oint_{b_i} \Omega_{X_l Y_l}, \quad S_{lr} = \int_{X_l}^{Y_l} \Omega_{X_r Y_r} = \int_{X_r}^{Y_r} \Omega_{X_l Y_l} = S_{rl} \ (l \neq r), \tag{2.8}$$
$$i = 1, \dots, g, \quad l, r = 1, \dots, s.$$

As follows from (2.8), in the limit $Y_1, \dots Y_s \to Q_1$, which transforms the extended map (1.1) to (1.2) with $n = s + 1$, the integrals S_{lr} become infinite and the theta function $\tilde{\theta}(\mathbf{z}, Z)$ becomes singular. Alternatively, one may try to divide the sum (2.7) by certain exponents of S_{lr} *before* taking the limit to obtain a finite expression instead. However, this procedure involves several delicate steps, which appeared to be more complicated than the direct approach proposed in this paper.

Our proof is inductive. In section 3 we will establish Theorem 2.2 for the case $n = 2$. Section 4 contains the inductive step and completes the proof of Theorem 2.2. Here we also establish the formulae (2.4) and give a proof of Theorem 2.3 closing with some remarks on the generalized theta divisor. We conclude the section by proving Proposition 2.1.

Proof of Proposition 2.1. 1). Our choice of normalization of the differentials ω_i and Ω_{i1} together with the invariance $\theta(\mathbf{z}+\mathbf{e}_k) = \theta(\mathbf{z})$, where $\mathbf{e}_k$ is the vector with 1 in the k-th entry and zero otherwise, says that $f(P)$ is invariant as P encircles an a-cycle. Similarly encircling a γ-cycle at most changes the exponential factors by $2\pi i$'s and we again have invariance in this case. Finally consider encircling a b-cycle, say b_k. The quasi-periodicity of the Riemann theta function

$$\theta\left(\mathbf{z} + \oint_{b_k} \omega;\, \tau\right) = \exp[-i\pi(\tau_{kk} + 2z_k)]\,\theta(\mathbf{z};\, \tau)$$

means that the j-th term of the sum on the right-hand side of (2.1) changes by the phase

$$2\pi i\left(\mathbf{z} - K - A(P) - \mathcal{S} + A(Q_j)\right)_k - \oint_{b_k} \Omega_{j1} - i\pi\tau_{kk}.$$

Now the bilinear relation

$$2\pi i(A(Q_j) - A(Q_1)) = 2\pi i \int_{Q_1}^{Q_j} \omega_k = \oint_{b_k} \Omega_{j1} \tag{2.9}$$

means this may be rewritten as

$$2\pi i\left(\mathbf{z} - K - A(P) - \mathcal{S} + A(Q_1)\right)_k - i\pi\tau_{kk} \tag{2.10}$$

and this is just the change of phase of the remaining term on the right-hand side of (2.1).

Therefore $\Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P))$ changes by the phase (2.10) and hence is a quasi-periodic function. The quasi-periodicity of $f(P)$ means that although the function is not single-valued on the curve Γ its zeros do not depend on the choice of path in the Abel map $\mathcal{A}(P)$.

2). Next let us consider the behaviour of $f(P)$ as $P \sim Q_j$ $(j = 1, \ldots, n)$ focusing initially on the case $j \geq 2$. The i-th term in the sum of on the right-hand side of (2.1) for $f(P)$ contains the exponential term $\exp(-\int_{P_0}^{P} \Omega_{i1})$. This is regular as $P \to Q_j$ for $j \neq i$ and so as $P \sim Q_j$

$$d \ln \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P)) \sim d \ln \left(\alpha \exp(-\int_{P_0}^{P} \Omega_{j1}) + \beta\right).$$

Generically the coefficient α is non-vanishing. Now as $P \sim Q_j$ (with local coordinates τ)

$$\exp\left(-\int_{P_0}^{P} \Omega_{j1}\right) \sim \exp\left(-\int^{\tau} \frac{dz}{z}\right) \sim \frac{1}{\tau}$$

and consequently

$$d \ln \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P)) \sim d \ln \left(\frac{\alpha\tau + \beta}{\tau}\right) \tag{2.11}$$

has residue -1 at Q_j $(j \geq 2)$. Finally, as $P \sim Q_1$ (with local coordinate τ) each of the exponential terms

$$\exp\left(-\int_{P_0}^{P} \Omega_{j1}\right) \sim \exp\left(\int^{\tau} \frac{dz}{z}\right) \sim \tau \tag{2.12}$$

give a regular vanishing and so as $P \sim Q_1$

$$d \ln \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P)) \sim d \ln(\alpha\tau + \beta). \tag{2.13}$$

Generically $\beta \neq 0$ and this is regular at Q_1 and hence has vanishing residue. Indeed, considering now all of (2.1), we find that

$$\lim_{P \to Q_1} \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P)) = -\theta\left(\mathbf{z} - K - \sum_{i=1}^{n} A(Q_i)\right) \equiv \beta. \tag{2.14}$$

Thus we have established that $d \ln \Theta(\hat{\mathbf{z}} - \mathcal{A}(P))$ is regular at Q_1 and has poles of residue -1 at each of $Q_2, \dots, Q_n$.

3). To count the number of zero's we evaluate

$$\frac{1}{2\pi i}\oint_{\partial\hat{\Gamma}_c} d \ln \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P)) = \frac{1}{2\pi i}\left[\sum_{k=1}^{g}\oint_{a_k + a_k^{-1} + b_k + b_k^{-1}} - \sum_{r=1}^{n}\int_{\gamma_r}\right] d \ln \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P))$$

(upon noting that we encircle the Q_i's in the opposite direction to the outer boundary). Now utilizing the residues just determined the second term in this expression contributes

$$-\frac{1}{2\pi i}\sum_{r=1}^{n}\int_{\gamma_r} d \ln \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P)) = -\sum_{P \in \{Q_1, \dots, Q_n\}} \operatorname{Res}_P d \ln \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P) = n - 1.$$

The first term on the other hand is simplified using the quasi-periodicity properties of part (1) of the proposition. Letting f^- denote the value of f on the cycles a_k^{-1} or b_k^{-1} we have that if P lies on a_k then from (2.10)

$$d \ln f^-(P) = d \ln f(P) - 2\pi i\, \omega_k(P)$$

while if P is on b_k then

$$f^-(P) = f(P).$$

Thus

$$\frac{1}{2\pi i}\oint_{a_k + a_k^{-1}} d \ln f(P) = \frac{1}{2\pi i}\oint_{a_k}\left[d \ln f(P) - d \ln f^-(P)\right] = \oint_{a_k} \omega_k(P) = 1$$

and

$$\frac{1}{2\pi i}\oint_{b_k + b_k^{-1}} d \ln f(P) = 0.$$

Together these yield

$$g = \frac{1}{2\pi i}\sum_{k=1}^{g}\oint_{a_k + a_k^{-1} + b_k + b_k^{-1}} d \ln \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P)$$

and so the number of zeros of $f(P)$ is then $g + n - 1$. □

3. Clebsch's Case: $n = 2$

We now make the first step in the proof of Theorem 2.2. Namely, following [CG86], consider the extended Abel map $\mathcal{A}(P_1, \dots, P_{g+1}) \mapsto \mathbb{C}^{g+1} = \hat{\mathbf{z}} = (\mathbf{z}, Z_2)$, defined by

$$\int_{P_0}^{P_1} \omega + \cdots + \int_{P_0}^{P_{g+1}} \omega = \mathbf{z}, \tag{3.1}$$

$$\int_{P_0}^{P_1} \Omega_{21} + \cdots + \int_{P_0}^{P_{g+1}} \Omega_{21} = Z_2, \tag{3.2}$$

where $P_0 \neq Q_{1,2}$. We are going to show that its inversion is given in terms of the generalized theta function that comes from (2.1), (2.2),

$$\begin{aligned}\Theta_2(\hat{\mathbf{z}} - \mathcal{A}(P)) = \exp\left(Z_2 - \mathcal{K}_2 - \int_{P_0}^{P} \Omega_{21} \right) \theta\left(\mathbf{z} - K - A(P) - A(Q_1)\right) \\ - \theta\left(\mathbf{z} - K - A(P) - A(Q_2)\right),\end{aligned} \tag{3.3}$$

where

$$\mathcal{K}_2 = \sum_{k=1}^{g} \oint_{a_k} \omega_k(P) \int_{P_0}^{P} \Omega_{21} + \oint_{b_k} \Omega_{21}. \tag{3.4}$$

That is, we now establish the $n = 2$ case of Theorem 2.2.

Proposition 3.1. *Let $\hat{\mathbf{z}}$ be the right hand side of the extended map (3.1), (3.2) such that $f(P) = \Theta_2(\hat{\mathbf{z}} - \mathcal{A}(P))$ does not vanish identically on $\hat{\Gamma} \setminus \{Q_1, Q_2\}$. Then $f(P)$ has precisely $g+1$ zeros (possibly with multiplicity) giving a unique solution to the inversion of the map.*

Proof. The number of zeros and quasi-periodicity of $f(P)$ have already been described in Proposition 2.1 and it remains to show that this function uniquely solves the inversion.

First let us suppose $P_{g+1} = P_0$. Then (3.1) becomes the usual Abel-Jacobi map and $\mathbf{z}$ determines the divisor $P_1 + \cdots + P_g$ uniquely. In this case Z_2 becomes a function of $\mathbf{z}$. Following [CG86], we express (3.2) as a sum of residues and obtain

$$\begin{aligned} Z_2 &= \int_{P_0}^{P_1} \Omega_{21} + \cdots + \int_{P_0}^{P_g} \Omega_{21} = \sum_{k=1}^{g} \mathrm{Res}_{P=P_k} \left(\int_{P_0}^{P} \Omega_{21} \right) d \ln \theta(\mathbf{z} - K - A(P)) \\ &= \frac{1}{2\pi i} \oint_{\partial \hat{\Gamma}_c} \left(\int_{P_0}^{P} \Omega_{21} \right) d \ln \theta(\mathbf{z} - K - A(P)) \\ &= \frac{1}{2\pi i} \oint_{\partial \hat{\Gamma}} \left(\int_{P_0}^{P} \Omega_{21} \right) d \ln \theta(\mathbf{z} - K - A(P)) - \mathrm{Res}_{P \in \{Q_1, Q_2\}} \left(\int_{P_0}^{P} \Omega_{21} \right) d \ln \theta(\mathbf{z} - K - A(P)) \\ &= \frac{1}{2\pi i} \oint_{\partial \hat{\Gamma}} \left(\int_{P_0}^{P} \Omega_{21} \right) d \ln \theta(\mathbf{z} - K - A(P)) + \mathrm{Res}_{P \in \{Q_1, Q_2\}} \Omega_{21}(P) \ln \theta(\mathbf{z} - K - A(P)) \\ &= \hat{\mathcal{K}}_2 + \ln \frac{\theta(\mathbf{z} - K - A(Q_2))}{\theta(\mathbf{z} - K - A(Q_1))}, \end{aligned} \tag{3.5}$$

where we set

$$\hat{\mathcal{K}}_2 = \frac{1}{2\pi i} \oint_{\partial \hat{\Gamma}} \left(\int_{P_0}^{P} \Omega_{21} \right) d \ln \theta(\mathbf{z} - K - A(P)) = \frac{1}{2\pi i} \sum_{k=1}^{g} \oint_{a_k + a_k^{-1} + b_k + b_k^{-1}} h(P) dg(P). \tag{3.6}$$

Here

$$h(P) = \int_{P_0}^{P} \Omega_{21}, \quad g(P) = \ln \theta(\mathbf{z} - K - A(P))$$

and in the derivation of (3.5) we have used the single valuedness of $h(P)$ and $g(P)$ on $\hat{\Gamma}_c$.

Now let us consider in (3.1), (3.2) the generic positive divisor $P_1 + \cdots + P_{g+1}$ where $P_j \neq Q_1, Q_2$. This situation can be reduced to the previous one by replacing $\hat{\mathbf{z}}$ by $\hat{\mathbf{z}} - \mathcal{A}(P_j)$.

Then the relation (3.5) gives rise to

$$\exp\left(Z_2-\hat{\mathcal{K}}_2-\int_{P_0}^{P_j}\Omega_{21}\right)=\frac{\theta(\mathbf{z}-K-A(Q_2)-A(P_j))}{\theta(\mathbf{z}-K-A(Q_1)-A(P_j))},\qquad j=1,\dots,g+1.$$

That is, if $\hat{\mathcal{K}}_2=\mathcal{K}_2$, the function $f(P)=\Theta_2(\hat{\mathbf{z}}-\mathcal{A}(P))$ vanishes at $P_1,\dots,P_{g+1}$ with the same multiplicity. According to Proposition 2.1, in our case $f(P)$ has precisely $g+1$ zeros, which therefore must coincide with $P_1,\dots,P_{g+1}$. This proves Proposition 3.1.

It remains only to show that $\hat{\mathcal{K}}_2$ coincides with $\mathcal{K}_2$ given by expression (3.4). Let h^-, g^- denote the value of h, respectively g on the cycles a_k^{-1} or b_k^{-1}. Then, if P lies on a_k,

$$h^-(P)=h(P)+\oint_{b_k}\Omega_{21},\qquad g^-(P)=g(P)-2\pi i\,\omega_k(P)$$

while if P is on b_k then

$$h^-(P)=h(P),\qquad g^-(P)=g(P).$$

Thus

$$\begin{aligned}\oint_{a_k+a_k^{-1}}h(P)dg(P)&=\oint_{a_k}\left[h(P)dg(P)-h^-(P)dg^-(P)\right]\\&=2\pi i\oint_{a_k}\omega_k(P)\int_{P_0}^{P}\Omega_{21}-\oint_{b_k}\Omega_{21}\oint_{a_k}dg(P)+2\pi i\oint_{b_k}\Omega_{21}\\&=2\pi i\oint_{a_k}\omega_k(P)\int_{P_0}^{P}\Omega_{21}+2\pi i\oint_{b_k}\Omega_{21},\end{aligned}$$

as the periodicity by shifts of a cycles means $\oint_{a_k}dg(P)=0$. Similarly we have that

$$\oint_{b_k+b_k^{-1}}h(P)dg(P)=0$$

and therefore (3.6) yields the expression (3.4) . ☐

4. The General n case

We next inductively establish the general case of Theorem 2.2. We assume Theorem 2.2 holds for $n-1$ and consider

$$\begin{array}{rcl}\displaystyle\int_{P_0}^{P_1}\omega+\cdots+\int_{P_0}^{P_{g+n-1}}\omega&=&\mathbf{z},\\ \displaystyle\int_{P_0}^{P_1}\Omega_{21}+\cdots+\int_{P_0}^{P_{g+n-1}}\Omega_{21}&=&Z_2\\ \vdots&&\vdots\\ \displaystyle\int_{P_0}^{P_1}\Omega_{n-1,1}+\cdots+\int_{P_0}^{P_{g+n-1}}\Omega_{n-1,1}&=&Z_{n-1}\\ \displaystyle\int_{P_0}^{P_1}\Omega_{n1}+\cdots+\int_{P_0}^{P_{g+n-1}}\Omega_{n1}&=&Z_n,\end{array}\tag{4.1}$$

where $P_0\neq\{Q_1,\dots,Q_n\}$. We wish to show that the inversion of this map is given by zeros of the generalized theta function Θ_n that comes from (2.1), (2.2).

We again begin by supposing that $P_{g+n-1} = P_0$. Then, using the inductive hypothesis, the solutions of the first $n-2$ of equations (4.1) are given in terms of $\Theta_{n-1}(\mathbf{z}, Z_2, \dots, Z_{n-1})$, and in this case Z_n is a function of $\hat{\mathbf{z}} = (\mathbf{z}, Z_2, \dots, Z_{n-1})$, namely, the sum of residues,

$$\begin{aligned} Z_n &= \int_{P_0}^{P_1} \Omega_{n1} + \cdots + \int_{P_0}^{P_{g+n-2}} \Omega_{n1} \\ &= \sum_{k=1}^{g+n-1} \operatorname{Res}_{P=P_k} \left(\int_{P_0}^{P} \Omega_{n1} \right) d \ln \Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(P)) \\ &= \frac{1}{2\pi i} \oint_{\partial \hat{\Gamma}_c} \left(\int_{P_0}^{P} \Omega_{n1} \right) d \ln \Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(P)). \end{aligned}$$

Noting that the functions $\Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(P))$, $\int_{P_0}^{P} \Omega_{n1}$ are single valued on $\hat{\Gamma}_c$ and taking orientations into account, the latter expression may be rewritten as

$$\begin{aligned} Z_n &= \frac{1}{2\pi i} \left[\oint_{\partial \hat{\Gamma}} - \int_{\gamma_1 + \cdots + \gamma_n} \right] \left(\int_{P_0}^{P} \Omega_{n1} \right) d \ln \Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(P)) \\ &= \mathcal{K}_n - \frac{1}{2\pi i} \int_{\gamma_1 + \cdots + \gamma_n} \left(\int_{P_0}^{P} \Omega_{n1} \right) d \ln \Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(P)), \end{aligned}$$

where we set

$$\mathcal{K}_n = \frac{1}{2\pi i} \oint_{\partial \hat{\Gamma}} \left(\int_{P_0}^{P} \Omega_{n1} \right) d \ln \Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(P)). \tag{4.2}$$

According to item 2) of Proposition 2.1, for generic $\hat{\mathbf{z}}$ the differential $d \ln \Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(P))$ has a simple pole (of residue -1) at $Q_2, \dots, Q_{n-2}$ and is regular at $Q_{1,n}$ while Ω_{n1} has simple poles at $Q_{1,n}$. Upon an integration by parts we get

$$\begin{aligned} Z_n &= \mathcal{K}_n + \int_{P_0}^{Q_2} \Omega_{n1}(P) + \cdots + \int_{P_0}^{Q_{n-1}} \Omega_{n1}(P) + \frac{1}{2\pi i} \oint_{\gamma_1 + \gamma_n} \Omega_{n1}(P) \ln \Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(P)) \\ &= \mathcal{K}_n + \int_{P_0}^{Q_2} \Omega_{n1}(P) + \cdots + \int_{P_0}^{Q_{n-1}} \Omega_{n1}(P) + \ln \frac{\Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(Q_n))}{\Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(Q_1))}. \end{aligned}$$

Next, in view of the property (2.14), the above relation yields

$$\exp \left(Z_n - \mathcal{K}_n - \sum_{k=2}^{n-1} \int_{P_0}^{Q_k} \Omega_{n1}(P) \right) \theta \left(\mathbf{z} - K - \sum_{i=1}^{n-1} A(Q_i) \right) + \Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(Q_n)) = 0,$$

which, in view of the recursive definition (2.2), is equivalent to $\Theta_n(\mathbf{z}, Z_2, \dots, Z_n) = 0$.

Now consider in (4.1) the generic positive divisor $P_1, \dots, P_{g+n-1}$ with $P_j \neq Q_1, \dots, Q_n$. Then upon replacing $\hat{\mathbf{z}}$ by $\hat{\mathbf{z}} - \mathcal{A}(P_j)$ (for any $j \in \{1, \dots, g+n-1\}$) we reduce to the case just considered and have that $\Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P_j)) = 0$. On the other hand, in view of Proposition 2.1, the function $\Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P))$ has precisely $g+n-1$ zeros, which therefore coincide with $P_1, \dots, P_{g+n-1}$. This completes the proof of Theorem 2.2.

It remains to derive the expression (2.4) for $\mathcal{K}_i$. Our induction will have established this once we show that $\mathcal{K}_n$ has this form. Rewriting (4.2) we have that

$$\mathcal{K}_n = \frac{1}{2\pi i} \sum_{k=1}^{g} \oint_{a_k + a_k^{-1} + b_k + b_k^{-1}} h(P)\, dg(P)$$

where $h(P) \equiv \int_{P_0}^{P} \Omega_{n1}$ and $g(P) \equiv \ln \Theta_{n-1}(\hat{\mathbf{z}} - \mathcal{A}(P))$ are single valued on $\hat{\Gamma}_c$. Again letting h^- denote the value of h on the cycles a_k^{-1} or b_k^{-1} (and similarly for g) we have that if P lies on a_k then

$$h^-(P) = h(P) + \oint_{b_k} \Omega_{n1}, \qquad g^-(P) = g(P) - 2\pi i\, \omega_k(P)$$

while if b_k then $h^-(P) = h(P)$, $g^-(P) = g(P)$. Exactly paralleling the $n = 2$ calculation we have that

$$\oint_{a_k + a_k^{-1}} h(P) dg(P) = 2\pi i \oint_{a_k} \omega_k(P) \int_{P_0}^{P} \Omega_{n1} + 2\pi i \oint_{b_k} \Omega_{n1},$$

and

$$\oint_{b_k + b_k^{-1}} h(P) dg(P) = 0.$$

Therefore

$$\mathcal{K}_n = \sum_{k=1}^{g} \oint_{a_k} \omega_k(P) \int_{P_0}^{P} \Omega_{n1} + \oint_{b_k} \Omega_{n1},$$

as required.

Proof of Theorem 2.3. Having the proof of Theorem 2.2, we are able to prove Theorem 2.3. It assumes that $f(P) = \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P))$ does not vanish identically. Hence, $f(P)$ has precisely $g + n - 1$ zeros $P_1 + \cdots + P_{g+n-1}$. On the other hand, since $\hat{\mathbf{z}} \in \{\Theta_n(\hat{\mathbf{z}}) = 0\}$, we have that $f(P_0) = 0$, and one of the above zeros, say P_{g+n-1}, coincides with P_0. Then, by Theorem 2.2, there is the unique representation

$$\hat{\mathbf{z}} = \mathcal{A}(P_1) + \cdots + \mathcal{A}(P_{g+n-2}) + \mathcal{A}(P_0),$$

which is equivalent to (2.6). □

Remark 2. Both the theta divisor Θ and the set $\{e\} \subset \Theta \subset \mathrm{Jac}(\Gamma)$ for which the function $f(P) = \theta(e - A(P))$ vanishes identically are interesting and much studied objects [FK80, Ch. VI §3]. Analogous to these we have the generalized (translated) theta divisor

$$\Xi = \{\Theta_n(\hat{\mathbf{z}}) = 0\} \subset \mathrm{Jac}(\Gamma; Q_1; Q_2, \ldots, Q_n)$$

and the set $\mathcal{B} \equiv \{e\} \subset \mathrm{Jac}(\Gamma; Q_1; Q_2, \ldots, Q_n)$ for which $f(P) = \Theta(e - \mathcal{A}(P))$ *does* vanish identically. Again $\mathcal{B} \subset \Xi$, for if e does not belong to Ξ then $f(P)$ cannot vanish identically. Although our inversion theorems have focused above on the subset $\{\hat{\mathbf{z}}\} \subset \mathrm{Jac}(\Gamma; Q_1; Q_2, \ldots, Q_n)$ for which $f(P) = \Theta_n(\hat{\mathbf{z}} - \mathcal{A}(P))$ does not vanish identically, both Ξ and $\mathcal{B}$ nonetheless appear interesting objects of study.

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